Games, Numbers and Conway

An introduction to the strange games and numbers invented by the remarkable contemporary mathematician,

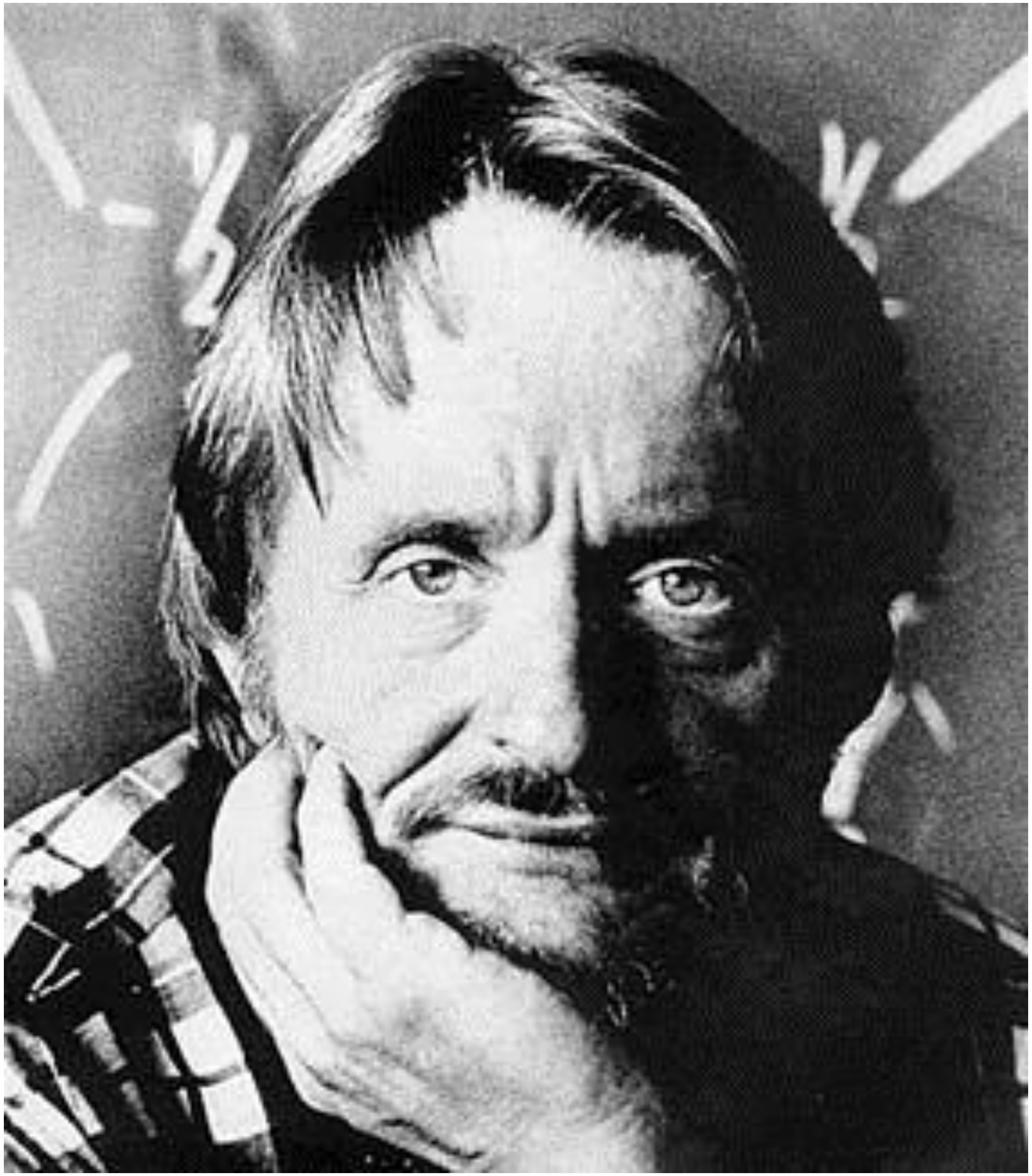
John Horton Conway

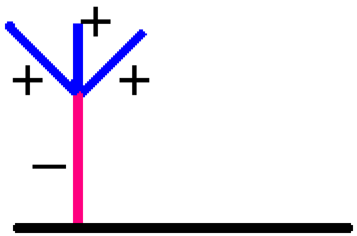
A talk sponsored by the SJFC mathematics program as part of Math Awareness Month presented by

Dr. Gerry Wildenberg

Among Conway's achievements:

- Conway group (Group theory)
- Sphere packing theorems
- Knot theory
- Life (Cellular Automata – Game of Life)
- Error correcting codes
- Quadratic forms
- The Conway numbers (a/k/a Surreal Numbers)
- Combinatorial game theory



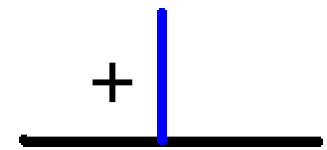


Red Blue Hackenbush

(Sample position on left)

Rules

1. Two players: bLue (Left) **R**ed (Right)
2. Left can remove a blue segment, Right can remove a red segment.
3. Any segments not connected to the ground disappear.
4. First player who can not move LOSES.



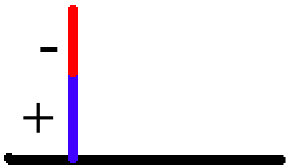
In first position, whoever is on move loses. We'll call this position 0.

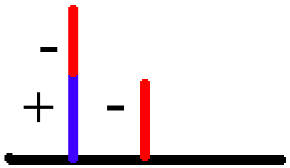
In the second position, if it's Red's turn, Red loses and if it's Blue's turn, Blue uses the free move available to Blue and again Blue wins. We'll call this +1.

You can probably guess what position represents –1.

Since you may not have red and blue (and later green) pens handy, I'm labeling the segments + or -.

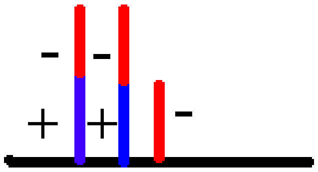
Who wins the position to the right?





Since Blue wins, the above is positive. Now look at the same position with an extra move for red. Who wins the position to the left?

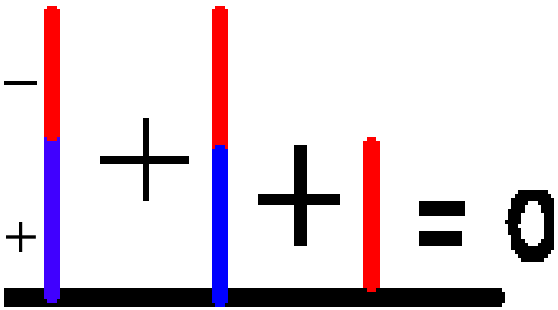
From these we see that the first position is an advantage for Blue but less than 1 move!! Now consider the position on the right. Who wins?



Choices: a. First b. Second c. Blue d. Red

Since whoever moves first LOSES, this is 0.

Since the first player to move LOSES this is equivalent to 0!! Thus we've shown that:

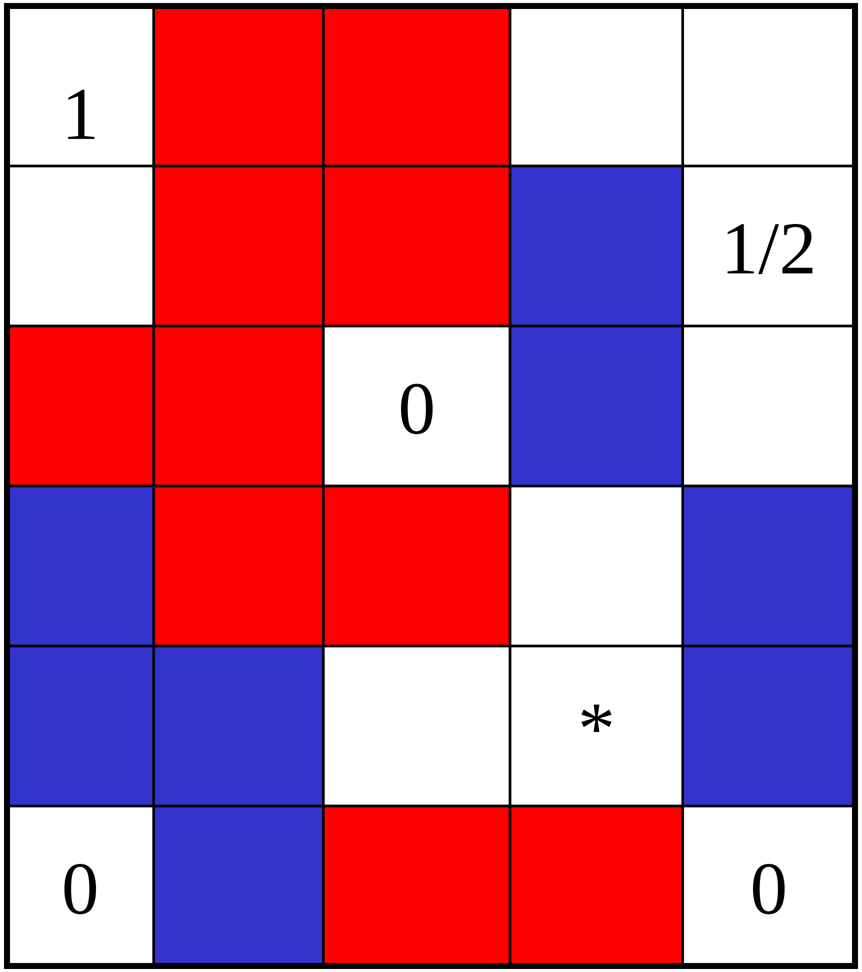


If we think of Left as positive, then we can think of this as: **1/2 + 1/2 + (–1) = 0**

This last example illustrates the idea of adding games. One of Conway's goals was to find numbers that represented games and could be added. In that way, complicated games can be analysed as a combination of many simpler games.

Note: We play a sum of games by playing in any one of the components.

This game illustrates the way a game can become separated into smaller regions which can be analysed.



In the game called "Domineering", Blue inserts horizontal dominoes covering two squares and Red vertical dominoes. Last player to move wins.

(We'll see * later.)

Numbers and games.

We can represent game positions in terms of the choices for each player.

So the zero position is represented: { | } since in the zero position neither left nor right can move.

{0 | } is the position where Right has no moves but Left can move to the 0 position. This is the position of 1 Blue line and hence is +1.

{ | 0} is the position where Left has no moves but Right can move to the 0 position. This is the position of 1 red line and hence is –1.

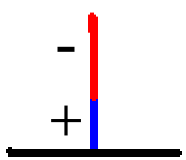
(From now on I'll often identify the position and the number so I could just say: "Left can move to 0").

Now I'll claim that n+1={n| }. Why is this?

The reason that n+1 = { n | } is that in a position where left has n+1 free moves she can move to a position where she has n moves remaining.

We can also show that {n|n+1} = n + 1/2

As an example recall our earlier claim that:



is just 1/2.

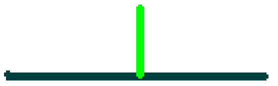
Note that Left can move to the zero position and Right can move to a position with 1 free move for Left. 1/2={0 | 1}

Now we enter a world of weirdness!

We’ve seen positions where the first player to move loses. (E.g. the basic zero position.) We’ve seen positions where Blue wins regardless of who starts (these have positive values) and similarly for Red (negative values). But we haven’t yet seen positions where the first player to move wins. We’ll see those now and we’ll call those positions “fuzzy” (I.e. not zero, not positive, not negative).

We’ll add a color to the game, Green. A green line can be removed by either player.

Conway calls this position ✱ (star). What number represents this position?

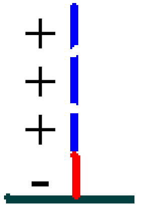


It’s not zero. It can be written: { 0 | 0 } (Why?).

This suggests that it’s between 0 and 0.

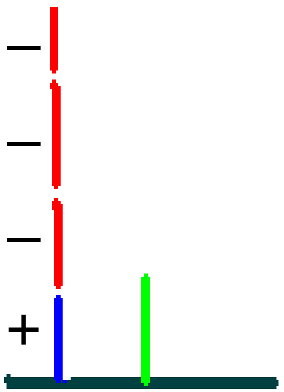
In fact, ✱ is not an ordinary number. We’ll see in a moment that * is less than every positive number and greater than every negative number.

What value has this number? Similarly to our evaluation of 1/2 this turns out to be –1/8 . And it’s easy to guess what +1/8 will be.



Now we can look at the sum on the right. If Right moves, she can take any red or the green. If she takes the green, then Left wins immediately, so we'll start with Right taking the top red. Then Left will take the green and then Right takes a 2nd red. Then Left will win by taking the blue. If Left moves first, then she takes the green and soon wins.

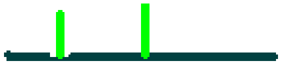
In fact, ANY positive value for the red&blue part, no matter how tiny, will make this a win for Left. Similarly, if the red&blue part is negative, this is a win for Right.



Notation:

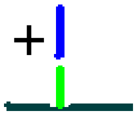
Conway calls pictures like the red-blue above: "Hollyhocks". The man has a gift for colorful notation.

Try now to evaluate * + *.

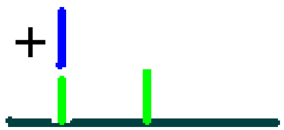


If you got 0, you are right.
But you may be beginning to doubt my claim that * ≠ 0

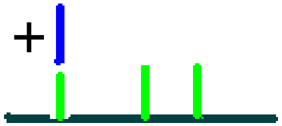
The picture to the right is easily seen to be fuzzy as whoever moves first takes the whole stalk and wins. Yet we have the uneasy feeling that it should favor Blue who has an extra move.

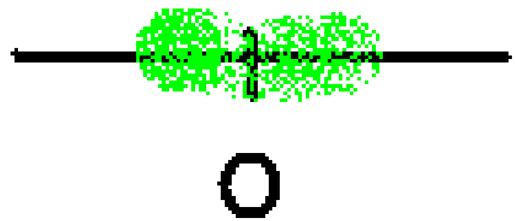


Here's the previous picture with one extra green stalk. What's this one's value? That's not obvious. However it's easy to see this is positive (blue wins). But you can also show that this is less than any ordinary positive number!!



Furthermore, if we add another * we get this position which is fuzzy! In other words the above diagram is so weakly positive that just the addition of one green line makes the result fuzzy.





Conway pictures these small “fuzzy” numbers as living in a cloud around 0. Unlike the Reals where we can always compare two numbers, in Conway’s world, there are numbers that can’t be compared. For example,0 and * can’t be compared.

Bibliography

1. Winning Ways by Berlekamp, Conway and Guy. (The Bible.)
2. On Numbers and Games (Often referred to as ONAG) by J. H. Conway
3. Mathematical Games by Beasley (Much less complete than the above.)
4. Surreal Numbers by Donald Knuth (A great introduction to the Conway numbers but nothing about games.)

Rules for Life

- By a “neighbor” I mean an adjacent, populated cell
- Occupied cells with 0 or 1 neighbor(s) die of exposure
- Occupied cells with 2 or 3 neighbors survive
- Occupied cells with 4 or more neighbors die of overcrowding
- Empty cells with 3 neighbors become populated – this is the only way cells become populated
- All change occurs simultaneously based on the previous generation