3-Adding Games

A key concept in the Theory of Games is "adding" games.

The definition of the sum of two (or more) games is not too difficult but is a little strange. Just as the sum of two numbers is another number, the sum of two games is another game. The rules for the new game are these:

- A move in the sum is a move in either (any) of the original games.
- As usual, the winner of the sum is the last player to make a move in any of the original games.

To see why this may be interesting, let's look at Nim.

1-pile-Nim is a game that is so simple that at first it will seem pointless. Please withhold judgment. 1-pile-Nim is just like Take-1-2-3 except that the player who moves can take *any* number of matches from the pile. The exact number in the pile needs to be determined by the players before the start of the game. This game is completely trivial since in any size pile, the first player to move can just take the whole pile and thus win.

Let's look at a sample game of the sum of two games of 1-pile-Nim. We can call the sum: 2-pile-Nim.

Pile 1	Pile 2	Player 1 action	Player 2 action	Comments
19	23	-4 from Pile 2		The only move to force a win
19	19		-9 from Pile 2	
19	10	-9 from Pile 1		
10	10		-6 from Pile 1	
4	10	-6 from Pile 2		Note how player 1 is forcing a win by keeping
				the piles equal
4	4		-3 from Pile 2	Player 2 is helpless
4	1	-3 from Pile 1		
1	1		-1 from Pile 1	
0	1	-1 from Pile 2		And player 1 has made the last move and wins

So we see that 2-pile-Nim is completely "solved" by the following two rules

- If the two piles are unequal, then Player 1 can force a win by equalizing the piles whenever it is their turn
- If the piles are equal at the start, then Player 1 is up the creek. Player 1 must unequalize and then Player 2 can follow the same strategy as above.

Nim (without the number of piles specified) is usually played as a sum of three games of 1-pile-Nim. Thus a sample game of Nim might begin with a position such as:



where this time we have pictures of the toothpicks.

Play this game now with your neighbor. Try to decide if you want to go first or second. This is a good time to recall rule 4 from the handout on basics. Either the first player or the second player has a forced win. (Note that we showed this for 1-pile-Nim and 2-pile-Nim

If you're sure you have 1,2,3 analyzed correctly, try again with some larger numbers like 2, 3 and 5

Summary of the Nim discussion so far:

1-pile-Nim is trivial.

2-pile-Nim is very simple

3-pile-Nim is difficult – it is not in the least obvious how to determine who wins a game of 3-pile-Nim.

We'll spend some time seeing if we can develop some rules to help us play this game. After a while we'll have an exact winning strategy (when one exists).

4 Nimbers (preview)

In order to analyze complex games like multi pile Nim, we'll need to give numerical values to every game position. That's our goal for today (and for some time after).

Let's agree (for all the games we'll study) that a position where the player whose turn it is loses has the value 0 (zero). Thus we call a position where the player on move *loses* a "zero position". Notice that a game of 1-pile-Nim which begins with zero toothpicks in the pile is a loss for Player 1 and thus is a zero position.

This simple definition has lots of implications.

We know that a single Nim pile with one or more toothpicks, CAN NOT have the value zero since the player on move wins that position. Let's take the simplest approach by saying that a Nim pile has the same value as its size. (Later we'll have a more complex theory of games which justifies this valuation. For now let's just say we want to see where this approach leads to.)

Before we continue our analysis of Nim, I want to emphasize that the size of the pile is NOT the evaluation we will use for Take-1-2-3. Nor will pile size be a good method of evaluating most of the games we will study. Never-the-less the methods we use for Nim will be useful in many other games.

Now recall 2-pile-Nim. Recall that if we have two equal piles that the player on move loses. Thus these are zero positions. This says that 3+3=0, 7+7=0, 215+215=0, etc. **In general, n+n=0.**

And let's look at the game with three piles of 1, 2 and 3. Since you analyzed this to show that it's a loss for the first player, we can now say that 1,2,3 is a zero position. Thus:

1+2+3=0 1+2+3+3=3 (adding 3 to both sides) 1+2+(3+3)=3 1+2+0=3

1+2=3

Well that's no surprise but:

1+2+3=0

1+2+3+2=2 (adding 2 to both sides)

1+3+(2+2)=2

1+3+0=2

1+3=2

And similarly you can show that 2+3=1

Because these game evaluations are something like numbers (but the laws of arithmetic are very different) and are closely related to the game of Nim, we call them *Nimbers*.

Once we have a general rule for adding Nimbers, we'll have a complete solution for the game of Nim.

5-Nimbers (the nitty gritty)

In order to add Nimbers we need to write the Nimber as a sum of powers of 2 using the largest numbers possible and not repeating.

The first few powers of 2 are listed in this table.

Power of 2	2 ⁸	2 ⁷	2 ⁶	2 ⁵	2 ⁴	2 ³	2 ²	2 ¹	2 ⁰
Value	256	128	64	32	16	8	4	2	1

Suppose I want to express 10 as a sum of powers of 2 as in the first sentence of this handout. Here are two WRONG answers:

a) 10 = 4 + 4 + 2

b)
$$10 = 6 + 2 + 2$$

Here is the correct answer:

c) 10 = 8 + 2

You should be able to see why a) and b) are wrong and c) is right.

While you can probably do small numbers in your head, here's an example of a systematic way to do it.

Say we want to break down 58 into "Nimber form"

Left to do	Largest power of 2				
	that "fits"				
58	32				
58-32 = 26	16				
26 -16 = 10	8				
10-8 = 2	2				
2-2=0	None				

Once two (or more) Nimbers are expressed as sums of powers of 2, we can apply the following rules to add them.

- 1. n+n=0 I.e. Pairs of powers of two cancel
- 2. After applying rule 1, just add the rest using ordinary addition

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In these examples the "breakdowns" are shown in parentheses: 58 + 30 = (32 + 16 + 8 + 2) + (16 + 8 + 4 + 2) = 32 + 4 = 36 1 + 2 = (1) + (2) = 3 1 + 3 = (1) + (2 + 1) = 22 + 3 = (2) + (2 + 1) = 1

Note that we previously showed some of the above using the fact that we'd shown that 1+2+3=0

A larger example:

22 + 14 + 12 + 6 = (16 + 4 + 2) + (8 + 4 + 2) + (8 + 4) + (4 + 2) = 16 + 2 = 18

These additions can be summed up by this rule (which is equivalent to "cancelling pairs")

When computing a sum of Nimbers:

If a power of 2 appears an even number of times, it does **not** appear in the sum.

If a power of 2 appears an odd number of times, it does appear in the sum.

Important warning: DO NOT USE ORDINARY ARITHMETIC WITH NIMBERS (except as specified in Rule 2 above)

6 How to Play Nim Perfectly

The key to playing Nim perfectly is the following pair of facts (which I won't prove):

- 1. If a player is on move in a zero position (i.e. a position whose Nim-value is zero), any move will leave a non-zero position.
- 2. If a player is on move in a non-zero position, there will always be at least one move that will result in a zero position.

Apply these facts using this guideline:

Perfect play is to always put your opponent in a zero position if you can. But if you find yourself in a zero position, you're dead if your opponent is an expert player. If caught in a zero position, make any move and then pray your opponent makes a mistake.

Here's an example with a game that starts 3, 5, 11.

3+5+11=(2+1)+(4+1)+(8+2+1)=8+4+1=13

Notice that you take 5 away from the 11 pile, the new sum will be 3+5+6 and that 3+5+6 = (2+1) + (4+1) + (4+2) = 0

How did I know to take 5 from the 11 pile? First I noticed the 8 in 11=8+2+1. Since that's the only 8 in the game, it has to go. Then I noticed the three 1's. That means one of those has to go. Since I need to take from the third pile that means the third pile is going to end up even. Also I have to leave both a 4 and a 2 in the third pile (else the 2's or 4's won't all pair up). So I need the third pile to be less than 8, at least 6, and even. That complicated logic leads to: Change the 11 to a 6.

In practice you don't need to work out all that in advance. As soon as you see that the third pile has to be less than 8 the correct move will be found quickly by trying 7=4+2+1 and then 6=4+2.

In some positions there will be more than one winning move. For example:

13=8+4+1 11=8+2+1 7=4+2+1

Taking away 1 from *any* of the three piles will win.

12, 11,7 AND 13,10,7 AND 13,11,6 are *all* zero positions. Check this out right now. Make sure you understand why this is so.

But for any given pile there is at most one winning move. Try taking 2 or 3 or whatever you want to try away from any of the piles and see that it leaves a non-zero position.