Lemma 5.5 says that if $e=\tau_1\tau_2\tau_3 \ldots \tau_{n-1}\tau_n$, where each τ_i is a transposition then n is an even integer.

The proof is by induction. We use the fact that in trying to prove the case for k, we may assume the result is true for all lower values. In this proof we use the fact that k-2 even implies that k is even. But how do we get k-2 involved? We'll show that soon.

First let's note that $e=\tau_1$ is impossible since a transposition moves two elements and e moves none. Next look at $e=\tau_1\tau_2$ Here since 2 is even, we've verified the result for k=2.

Now we look at arbitrary k. $e=\tau_1\tau_2\tau_3 \ldots \tau_{k-1}\tau_k$. If we use τ_k= (ab) we see the following three possibilities for $\tau_{k-1}\tau_k$:

(cd)(ab), (ca)(ab), (ab)(ab)

That is because τ_{k-1} can match τ_k in 0, 1 or 2 elements. Notice that (ab)=(ba) so order within a given transposition doesn't matter.

Now (cd)(ab) = (ab)(cd) (Disjoint cycles commute.)
And: (ca)(ab) = (ab)(bc) (Just check that both sides equal: (bca))
And: (ab)(ab) = *e*

In the last case we've now reduced $e=\tau_1\tau_2\tau_3 \ldots \tau_{k-1}\tau_k$ to $e=\tau_1\tau_2\tau_3 \ldots \tau_{k-2}$
But, as mentioned earlier, we can assume k-2 is even. So k is even.

For the other two cases, using the equations above, we can now assume that the element a has its rightmost position in the k-1 transposition (not in the kth). We can repeat this argument as many times as necessary. Each time, either (ab)(ab) disappears (by cancellation) and we're done by induction or the last occurrence of element a moves left. Can it occur that element a is never part of a pair that cancels? NO! If this were to happen we would have that element a is now only in τ_1=(ab).

But *e* moves nothing and if a is only in τ_1=(ab) then a goes to b. That's impossible. So we'll always get cancellation sooner or later.

$\therefore$ K is even.
QED