

Homework due Thursday November 29.

Be sure you give clear, convincing answers to each "WHY?" And be sure to carefully prove each step.

1. Prove that if  $H$  is a subgroup of  $G$  then the mapping  $f: aH \rightarrow bH$  given by  $f(ah) = bh$  for all  $h$  in  $H$  is a 1-1, onto map. (This proves that all left cosets of a subgroup  $H$  are the same size.)
2. Find all the subgroups of  $S_3$ . Then find the left AND right coset decompositions of  $S_3$  for each of the subgroups. For each subgroup, check if the right and left cosets are the same or different and be sure to note that as part of your answer.
3. Let  $G$  be an abelian group of order 6. Show that  $G$  is the unique (up to isomorphism) cyclic group of order 6 by following this outline.
  - a. Determine the possible orders of elements by using Lagrange's theorem.
  - b. Show that if  $G$  has more than one element of order 2 that  $G$  would have a subgroup of order 4. (Why can't this happen?) Thus  $G$  has at most one element of order 2.
  - c. If we prove  $G$  has an element of order 6, we're done. WHY? So we can ignore the possibility of an element being of order 6 for the moment.
  - d. So now consider the order of the elements not of order 2. This gives us an element of order 3. Can all the non-identity elements be of order 3? NO!, WHY?
  - e. Thus we have an element  $x$  of order 2 and  $y$  of order 3. WHY?
  - f. Now show the order of  $xy$  must be 6. Then you're done. WHY?
4. Let  $G$  be any group ( $G$  can be finite or infinite). Let  $S$  be any nonempty subset of  $G$  ( $S$  can be finite or infinite). Let  $E$  be the set of all finite products of elements of  $S$  and inverses of elements of  $S$ . Prove  $E$  is a subgroup. (Note: I take finite to include 0, so that  $x^0 = e$  is in  $E$ )
5. Let  $G$  be an infinite cyclic group with  $G = \langle a \rangle$ . Let  $\mathbb{Z}$  denote the integers. Define  $f: \mathbb{Z} \rightarrow G$  by  $f(k) = a^k$ . Show that  $f$  is an isomorphism. This shows that all infinite cyclic groups are isomorphic. WHY?
6. Show that if  $[G:H]=2$  then  $aH=Ha$  for all  $a$  in  $G$ .
7. Prove that if  $G_1$  and  $G_2$  are isomorphic groups and  $f$  is an isomorphism and  $H$  is a subgroup of  $G_1$  then:
  - a. For any  $g$  in  $G_1$ ,  $f(g^{-1}) = f(g)^{-1}$
  - b.  $f(H)$  is a subgroup of  $G_2$BTW:  $f(H)$  is defined as:  $\{f(h) \mid h \text{ in } H\}$ . This notation is used in all areas of mathematics not just group theory.