Classroom exercise Symmetries of the Square

Goals (achieving all these might take several days of discussion): 1.See an algebraic system in a geometric context. 2.Develop axioms for a group. 3.Learn a new operation.

Where a large bold **STOP** appears we will make sure everybody is on the same page. I will interrupt your work and get students to summarize what we have learned. If you reach one of these points before we decide to stop, you may (and should) go on working on the next section.

Equipment: A small colored square with the corners labeled 1, 2, 3, 4 (consecutively) and a sheet of paper containing a square labeled 1, 2, 3, 4 (consecutively). See below.

Before starting write 1 "behind" the 1 on the colored square; i.e. on the other side of the paper. Do the same for all four corners -2 behind 2, 3 behind 3, 4 behind 4.



Guide for discovering symmetries:

The term "symmetries" (above)refers to the fact that mathematicians call the object we are examining today, the **Symmetries of the Square**. They describe ways in which a square is symmetric.

How many different ways of *placing* the square on the square on the sheet, *with* corners of the small square on corners of the sheet square, can you find?

What were the different placements of the small square on the sheet square that you found? If you haven't done so yet, make a list.

What notation did you devise for describing the different placements? How many did you find?

STOP

We next think of the different placements (mathematicians would say "symmetries") in terms of how you can **move** to the position from the "start" position.

The last sentence is important – please read it again. From now on we'll be referring to the placements by the *motions* needed to get to them.

We'll classify the different placements (i.e. symmetries) that you found. The classification will be based on how you can **move** to the position from the 1 matching 1, 2 matching 2, 3 matching 3, 4 matching 4 position. I'll call that placement the "identity". In terms of motions, it's the symmetry reached by no movement. (If this seems weird consider this: the identity symmetry is exactly analogous to the function f(x)=x. For this function no value is changed. For the identity symmetry, no corner position is changed.)

We want to think of the different placements of the square which you found in terms of the motions needed to get to them from the start position.

Let's call one set of symmetries the "flips". What do you think I mean by a "flip"?

And we can denote the flips by F1, F2, \dots . How many of your placements can be found by "flipping" the square from the identity position.

There are other ways to move the square so that corners end up in corner positions, namely the rotations. How many did you find? What were they? Do you think of them as clockwise rotation or counterclockwise? Does it matter whether you think of them as clockwise or counterclockwise? If you think it matters, state why. If you think it doesn't matter, state why.

Let's call these R1, R2, \ldots .

Lastly let's call the *identity*, e.

STOP

We come now to the key idea for this exercise: how can we develop an "operation" (a *binary* operation if we want to use careful language) on symmetries. If you've forgotten what a "binary operation" is, or perhaps have never seen it defined, here's a definition: A binary operation on a set S is a function from SxS into S. For example, addition is a binary operation in which given two numbers we get a third number; so (2,3) yields 5 under the binary operation addition.

So now we look at an operation which takes two symmetries and yields a third symmetry. Our operation will simply be to apply one motion after the other. Suppose we follow a flip by a rotation. (We'll see later that what we are doing is really a "composition of functions" something you have seen in Math 200 and in Calculus courses.)

The next few questions involve you performing two "symmetry motions". For most students, the easiest way to do these is to think in terms of the motions. E.g. You might say to yourself or to your partner "First I flip across the horizontal axis, then I rotate 90° "

If you haven't yet done so pick a flip and try following it by a rotation. For example try F1 followed by R1. What did you get? Let's explain that question. A flip followed by a rotation (without restarting) gives some move of the square. Is the move you found by flipping and rotating on the list you made up earlier? (If not, then you've just discovered a new move that was not on the list. Is that likely?)

Now starting from the base position, try the same rotation followed by the same flip. I.e. Do the rotation first. Was the result the same as doing the flip first? If not, you've just discovered a non-commutative operation, perhaps one you haven't thought about before. Try F1 followed by R1 and R1 followed by F1. Try F1 followed by R2 and R2 followed by F1. //

Try some other combinations. What is the result of doing two different flips? Does performing your F1 followed by your F2, yield something on the list of symmetries? If answer is yes, which one? How about F2 followed by F1? Is that the same as the opposite order?

What can you say about F1 followed by F1 again? How about R1 followed by R1? Other repetitions of the same symmetry?

STOP

Now that we've tried a lot of combinations of symmetries lets get a little more systematic. Can we make a table that would show how ALL the pairs of symmetries can be combined? What would be a good way to organize this table so that it would be compact, complete, and clear? Answer this question by writing out a well-organized table that shows all the ways that symmetries can be combined. Hint: Your table might be rather like a multiplication or addition table.

STOP

Now let's look at some properties of the table of combining symmetries that you found. What property does the element we called "e" have? (See also next question.)

Fill in this sentence: The element e has the property that $\dots = \dots = \dots = \dots$ for each x in the set of symmetries.

How many times does e appear in your table?. How many times in each row? How many times in each column? Your table allows you to examine x followed by y and y followed by x; for the case of x followed by y yields e, look at y followed by x. Can you find a way to summarize this property? (Hint: see next question.)

Side note: Did you notice I defined a*b as b followed by a. Maybe your reaction was: "Why not the other way? Is Dr. W. just being annoying or is there a reason?" OK, here's a reason. Symmetries can be viewed as functions – permutations are a special kind of function. So let's recall function notation. $f \circ g$, the composition of f and g, means function g is applied first and then function f is applied. I.e. function notation means 2nd function is applied first. So the definition above is exactly consistent with the function notation you already know.

STOP

Notice that each symmetry of the square can be described by a permutation of (1,2,3,4). Make sure you understand this perfectly before going on to the next question.

Is the converse true? I. e. Does each permutation of (1,2,3,4) give a symmetry of the square? Why or why not? Can you give a proof of your answer? Hint for proof: Use a counting argument.

STOP