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A few approaches to $lim_{x\to\infty}(1+1/x)^x$

First let's recall what taking the log of a function and then computing limits does.
$f(x) \to 0 \Leftrightarrow ln(f(x)) \to -\infty$.
$f(x) \to 1 \Leftrightarrow ln(f(x)) \to 0$.
$f(x) \to e \Leftrightarrow ln(f(x)) \to 1$.
Using this we consider:

$ln((1+1/x)^x) = x \cdot ln(1+1/x)$
We now let t=1/x and then we calculate:
$lim_{t\to 0}\frac{ln(1+t)}{t}$
It's easy to see that we can use $L'H\hat{o}pital'sRule$ to find that this limit approaches 1 and hence that:
$lim_{x\to\infty}(1+1/x)^x = e$
Alternatively, instead of using $L'H\hat{o}pital'sRule$, we can note that:

$ln(1+t) = t - t^2/2 + t^3/3 - t^4/4 \cdots$. Then it's not hard to show that:
$lim_{t\to 0}\frac{ln(1+t)}{t} = lim_{t\to 0}1 - t/2 + t^2/3 + \cdots = 1$
For a rather different approach, we can start with the sequence $\{(1+1/n)^n\}_{n=1}^{\infty}$. Using induction, we can prove that this sequence is increasing. And we can also prove that it is bounded above, say by 3. Then we can define e as the limit of the sequence. (I'm using the fact that a monotonic increasing sequence bounded above has a limit.) A good question is how we know that e defined by this sequence is the same e that we get by looking at the inverse of the ln function but let's leave that one for another time.

I'd like to show that:

Theorem: $f(x) = (1 + 1/x)^x$ is a monotonic increasing function.

Proof:

f(x) increasing $\Leftrightarrow$ ln(f(x)) is increasing. So we consider:

$g(x) = x \cdot ln(1 + 1/x)$.

Then $g'(x) = ln(1 + 1/x) - 1/(1 + x)$. We want to show that this is positive. I.e. We want to show that: $ln(1 + 1/x) > 1/(1 + x)$. Taking exponentials, that last inequality is equivalent to: $1 + 1/x > e^{\frac{1}{1+x}}$.

Let t=1/(1+x). Then:

$e^t = 1 + t + t^2/2! + \cdots < 1 + t + t^2 + \cdots = 1/(1 - t) = 1 + 1/x$.

$\square$

Be sure to check all steps!! (I got this proof from Herb Silverman, College of Charleston. I also proved the same inequality, using the ln(x) series, but this proof is nicer.)