Theorem: The order of a conjugacy class of some element is equal to the index of the centralizer of that element.

In symbols we say: $|Cl(a)| = [G : C_G(a)]$

Proof: Since $[G : C_G(a)]$ is the number of left cosets of $C_G(a)$, we want to define a 1-1, onto map between elements in $Cl(a)$ and left cosets of $C_G(a)$.

Let $H = C_G(a)$.

Since the elements of $Cl(a)$ are conjugates of $a$, we define $f$ by:

$f : xax^{-1} \mapsto xH$

This is 1-1 since if $f(x) = f(y)$ then $xH = yH$ and thus $xh = y$. But then $x^{-1}y \in H$ so that $x^{-1}y$ commutes with $a$ and therefore: $x^{-1}ya = ax^{-1}y$. From that we conclude that $yay^{-1} = xax^{-1}$ proving 1-1.

As for onto, just note that for any $x \in G$, $xax^{-1}$ is in $Cl(a)$. However we also need to show the map is “well-defined”. I.e. we need to show that if $xax^{-1}$ and $yay^{-1}$ are two descriptions of the same element in $Cl(a)$, then $f(x) = f(y)$. The calculation which shows this is just the reversal of the steps above. To be precise:

$xax^{-1} = yay^{-1} \Rightarrow ax^{-1}y = x^{-1}ya \Rightarrow x^{-1}y$ commutes with $a \Rightarrow x^{-1}y \in H \Rightarrow xH = yH$

(Make sure you understand this last step.)

QED

Note that in a finite group this implies that $|Cl(a)|$ divides $|G|$. This observation lets us prove the following result:

Theorem: If $|G| = p^m$ for $p$ a positive prime, then $Z(G)$ (the set of all elements that commute with all elements in the group) is non-trivial.

This result says that in a “p-group” (i.e. a group of order $p^m$, that there are at least p elements that commute with all the elements in the group. Contrast this with $S_n$ where the center is only the identity.

Proof: $G$ is the union of disjoint conjugacy classes. (Why?). Let’s divide those into two sets: the “trivial” conjugacy classes which contain only themselves and the nontrivial conjugacy classes whose order is $>1$. Note that an element has a conjugacy class of order 1 iff it belongs to the center. So we consider the following disjoint union:

$G = Z(G) \cup Cl(a) \cup Cl(b) \cup ... \cup Cl(h)$

Each conjugacy class has order which divides $G$ (i.e. $p'$). But they are nontrivial. So $p$ divides each of the orders of the nontrivial conjugacy classes. But also $p$ divides $|G|$. Hence $p$ divides $|Z(G)|$. Therefore $|Z(G)| \geq p$.

QED